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Cylindrical beam; linear crack, torsion, conform mapping function, stress concentration problems

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Torsion of a cylindrical beam having two linear cracks with different lengths

Abstract
In this study, torsion problem of a circular shaped plate having two cracks with different lengths was studied. The conform function was established and calculated for solving the problem. Elasticity theory has been applied to calculate the stresses emerging at the critical points of the plate. Determination and solution of stress concentration problems occurring around damaged holes and cracks in machine elements (such as shafts, axle shaft etc.) hold great importance. An accurate estimation of the stresses occurring around the damaged regions depends highly on determination of the stress distribution in these areas. Several researches have been performed to understand the stress concentration occurring in the vicinity of such damages. In the literature, these problems have been investigated for elastic areas, mostly having a simple geometry. In previous studies stress concentration was examined on simple polygon areas rather than the complex polygon vertices surrounded by lines, since no conform function was available to conform such complex areas. In this study, conform functions were used for the first time to solve problems involving complex geometric polygons.

1. Introduction
Distribution of the stress concentration at the indents of the cracks forming in different types of holes, is crucial for the safety calculations of machine element parts. These machine elements include the shafts, rods, plates, spherical surfaces and cylinders. Service life of these parts is calculated through precise analysis of the stress distribution at the damaged regions. Several studies are available regarding the investigation of such damages [1,2,3]. In these studies, analyses of simple form elastic regions are emphasized. For the regions including the lines with a complex form, particularly the corners are examined less thoroughly. This arises from the inadequacy of conform functions for such regions (linear cracks are also included). Such type of functions was initially described in [4,5].

2. Problem Definition and Solution
In this study, the subject of the torsion-induced-stress analysis is a finite beam with cross-sectional area S, radius R and two different linear cracks surrounded by L contour, as shown in Figure 1. The cracks are positioned on the O_x axis. The coordinates of these crack tips are e_1 and e_2. Coordinate origin is placed at center of L contour. This problem can be solved by defining the $\varphi(z)$ function which satisfies the boundary conditions.

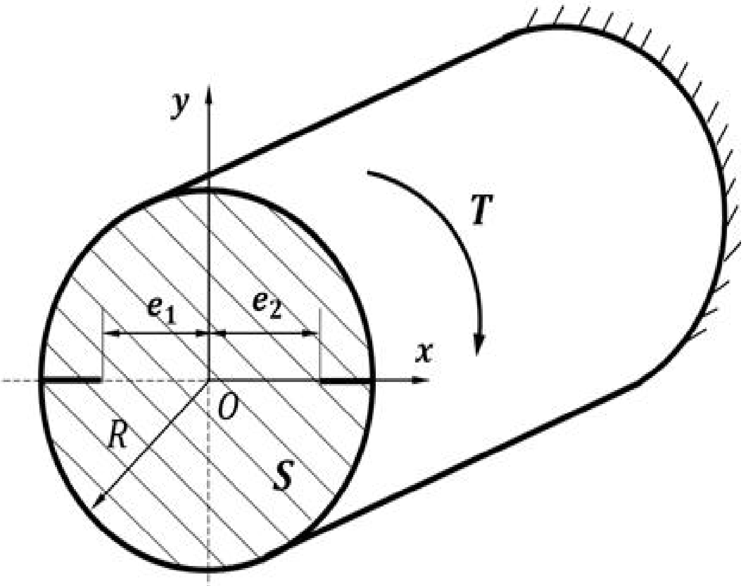

Figure 1. Cylindrical beam with two linear cracks.

$$\varphi(t)+\overline{\varphi(t)}=t.\bar{t}+c \tag{1}$$

The $\varphi(z)$ function, which is regular on any interior place of L contour, is defined as follows:

$$\varphi(z)=\sum_{k=1}^{\infty}A_k\left(\frac{z}{R}\right)^k+\alpha_0 \tag{2}$$

where

$$A_k=\sum_{n=0}^{\infty}{}^{*}\alpha_n * \beta^{(1)}_{\frac{n-k}{2}}$$

It is worth to consider that $\varphi(z)$ function is regular inside the L contour (for circle with radius R and two linear cracks). This function can be written using Fourier series as also shown in [4, 7]:

$$\varphi(t)=\sum_{n=-\infty}^{\infty}a_n[\chi(t)]^n \tag{3}$$

Here $\xi=\chi(t)$ is inverse function of conform function.

$$z=R.\xi.\sum_{n=0}^{\infty}\xi^{-2n}.\lambda_n \tag{4}$$

This inverse function was obtained as follows as also indicated in [4,7]:

$$\xi=\chi(z)=\frac{z}{R}\sum_{k=0}^{\infty}E_{k-1}.\left(\frac{R}{t}\right)^{2k} \tag{5}$$

If the lengths of two linear cracks are equal to each other ($e_1=e_2$) then conformal mapping functions (4) and (5) can be simplified properly. In this case, $a_1=b_1$ and $a=b$ are easily determined. It should be taken into account that if $e_1=R$ and $e_2=R$ are input in the shown functions, then the conform function of a radial cracked circumference will be found. Thus, $a=a_1=1$ or $b=b_1=1$ are considered. If the cracks do not exist (both of them is neglected, crack-free), i.e., if $e_1=R$ and $e_2=R$, then the formulas (4) and (5) can be transformed into the following simple forms.

$$\lambda_n = \sum_{k=n-2\varepsilon(\frac{n}{2})}^{n} \delta_k \cdot \gamma_{n-k}$$

$$E_n = \sum_{k=n-2\varepsilon(\frac{n}{2})}^{n} h_k \cdot g_{n-k}$$

$$\delta_n = \sum_{v=k-2E(\frac{k}{2})}^{k/2} \quad \sum_{n_1=n-2E(\frac{n}{2})}^{n} {}^{*}C^{n_1}_{-2v+1}\left[2\frac{b+a}{b-a}\right]^{n_1} .C^{\frac{n-n_1}{2}}_{-n}$$

$$\gamma_k = \sum_{v=k-2E(\frac{k}{2})}^{k/2} \varepsilon(-1)^v C^{v}_{1/2}\left(\frac{b-a}{4}\right)^{-2v+1} .C^{k-2v}_{-2v+1}$$

$$h_n = \sum_{n=v-2E(\frac{v}{2})}^{v/2} \sum_{n_1=k-2E(\frac{k}{2})}^{k} {}^{*}(-1)^{n_1} C^{n_1}_{-2n_1+1}(b+a)^{n_1} .C^{\frac{k-n_1}{2}}_{-n_1}$$

$$g_v = \sum_{n=v-2E(\frac{v}{2})}^{v/2} \varepsilon(-1)^n .C^{n}_{1/2} \cdot (b-a)^{2n-1} .C^{\frac{v-2n}{2}}_{-2n+1}$$

$$\varepsilon = \begin{Bmatrix} 1 \text{ if } n \neq 0 \\ 2 \text{ if } n = 0 \end{Bmatrix}$$

(6)

Both sides of Eq. (3) is multiplied by Cauchy kernel $[2\pi i(t-z)]^{-1}$ and integrated over L contour to obtain the value of $\varphi(t)$ function in the area of S, which is surrounded by L contour. In conclusion, the following result is obtained:

$$\frac{1}{2\pi i}\int_L \varphi(t).\frac{dt}{t-z} = \sum_{n=0}^{\infty}\frac{1}{2\pi i}\int_L a_k \frac{[\chi(t)]^n}{t-z}\,dt$$

Or

$$\varphi(z) = \sum_{n=0}^{\infty} a_k \frac{1}{2\pi i}\int_L \frac{[\chi(t)]^n}{t-z}\,dt \qquad (7)$$

During the integration process, it is observed that there is no integration for the $\chi(t)$ functions with negative exponent.

$$\frac{1}{2\pi i}\int_L \frac{[\chi(t)]^{-n}}{t-z}\,dt = 0 \qquad (8)$$

Finally the following expression was found for $\varphi(z)$ function by taking into account Equations (5) and (7).

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$$\varphi(z)=\sum_{n=0}^{\infty}a_k\frac{1}{2\pi i}\int_L\frac{[\chi(t)]^n}{t-z}dt=\sum_{n=0}^{\infty}a_k\frac{1}{2\pi i}\int_L\frac{\left[\frac{z}{A}\sum_{k=0}^{\varepsilon(n/2)}{}^{*}E_{k-1}(\frac{R}{z})^{2k}\right]^n}{t-z}dt$$

$$=\sum_{n=0}^{\infty}a_n\left(\frac{z}{R}\right)^n\sum_{k=0}^{\varepsilon(\frac{n}{2})}E_{k-1}^{(n)}\left(\frac{R}{z}\right)^{2k} \tag{9}$$

Here, the value of $\varepsilon\left(\frac{n}{2}\right)$ indicates that $\frac{n}{2}$ number is an integer.

When the addition sequence of two series in eq. (9) is changed, the following expression is obtained.

$$\varphi(z)=\sum_{\nu=0}^{\infty}A_\nu\left(\frac{z}{R}\right)^\nu \tag{10}$$

Here;

$$A_\nu=\sum_{n=\nu}^{\infty}{}^{*}a_n\beta_{\frac{n-\nu}{2}} \tag{11}$$

* sign shows that each subsequent term in the series increases by two.

All $E_\nu^{(n)}$ values are obtained in accordance with Recurrent formula [4, 7]:

$$E_\nu^{(n)}=\sum_{\nu_1=0}^{\nu}E_{k_1}^{(1)}E_{k-k_1}^{(n-1)} \tag{12}$$

Afterwards equation (10) is plugged in equation (1) at the boundary condition, and takes the following form:

$$\sum_{\nu=0}^{\infty}A_\nu\left(\frac{t}{R}\right)^\nu+\sum_{\nu=0}^{\infty}A_\nu\left(\frac{\bar{t}}{R}\right)^\nu=t.\bar{t}+C \tag{13}$$

By using conformal mapping function (4) the limits in (13) is transformed into the following formula:

$$\sum_{\nu=0}^{\infty}A_\nu\left[\tau\sum_{n=0}^{\infty}\lambda_n\xi^{-2n}\right]^\nu+\sum_{\nu=0}^{\infty}A_\nu\left[\tau^{-1}\sum_{n=0}^{\infty}\lambda_n\xi^{2n}\right]^\nu=R^2\sum_{n=0}^{\infty}\lambda_n\xi^{-2n}\sum_{k=0}^{\infty}\lambda_k\xi^{2k}+C \tag{14}$$

The above expression in the brackets can be written as follows:

$$\left[\tau\sum_{n=0}^{\infty}\lambda_n\xi^{-2n}\right]^\nu=\tau^\nu\left[\sum_{n=0}^{\infty}\lambda_n\xi^{-2n}\right]^\nu=\tau^\nu\sum_{n=0}^{\infty}\lambda_n^{(\nu)}\tau^{-2n}$$

$$\left[\tau^{-1}\sum_{n=0}^{\infty}\lambda_n\xi^{2n}\right]^\nu=\tau^\nu\sum_{n=0}^{\infty}\lambda_n^{(\nu)}\xi^{2n} \tag{15}$$

These expressions can be simplified as below:

$$\tau^\nu\sum_{n=0}^{\infty}\lambda_n^{(\nu)}\tau^{-2n}=\sum_{n=0}^{\infty}\lambda_n^{(\nu)}\tau^{-2n+\nu}=\sum_{k=-\nu}^{\infty}\lambda_{\frac{k+\nu}{2}}^{(\nu)}\tau^{-k}+\sum_{k=\nu}^{\infty}\lambda_{\frac{k+\nu}{2}}^{(\nu)}\tau^{-k}+\sum_{k=1}^{\infty}\lambda_{\frac{k+\nu}{2}}^{(\nu)}\tau^{-k}$$

$$=\sum_{k=0}^{\nu}\lambda_{\frac{\nu-k}{2}}^{(\nu)}\tau^{k}+\sum_{k=1}^{\infty}\lambda_{\frac{k+\nu}{2}}^{(\nu)}\tau^{-k} \tag{16}$$

These can be analogically written as:

$$\tau^{-\nu}\sum_{n=0}^{\infty}\lambda_n^{(\nu)}\tau^{2n}=\sum_{n=0}^{\infty}\lambda_n^{(\nu)}\tau^{2n-\nu}=\sum_{k=-\nu}^{\infty}\lambda_{\frac{k+\nu}{2}}^{(\nu)}\tau^{k}$$

$$=\sum_{k=-\nu}^{0}\lambda_{\frac{k+\nu}{2}}^{(\nu)}\tau^{k}+\sum_{k=1}^{\infty}\lambda_{\frac{k+\nu}{2}}^{(\nu)}\tau^{k}=\sum_{k=0}^{\nu}\lambda_{\frac{\nu-k}{2}}^{(\nu)}\tau^{-k}+\sum_{k=1}^{\infty}\lambda_{\frac{k+\nu}{2}}^{(\nu)}\tau^{k} \tag{17}$$

The right side of Equation (14) can be written follows:

$$R^2\sum_{n=0}^{\infty}\lambda_n\xi^{-2n}\sum_{k=0}^{\infty}\lambda_k\xi^{2k}=R^2\sum_{n=0}^{\infty}\lambda_n\sum_{k=0}^{\infty}\lambda_k\xi^{2k-2n}$$

$$=R^2\sum_{n=0}^{\infty}\lambda_n\sum_{\nu=0}^{n}\lambda_{n-\nu}\xi^{-2\nu}+R^2\sum_{n=0}^{\infty}\lambda_n\sum_{\nu=1}^{\infty}\lambda_{\nu+n}\xi^{2\nu} \tag{18}$$

When the equations 15, 16, 17, 18 are written considering the boundary conditions of the equation (14) then the following is obtained:

$$\sum_{\nu=0}^{\infty}A_\nu[\tau^{\nu}\sum_{n=0}^{\infty}\lambda_n^{(\nu)}.\tau^{-2n}]+\sum_{\nu=0}^{\infty}A_\nu[\tau^{-\nu}\sum_{n=0}^{\infty}\lambda_n^{(\nu)}.\xi^{2n}]$$

$$=R^2\sum_{n=0}^{\infty}\lambda_n\sum_{\nu=0}^{n}\lambda_{n-\nu}.\xi^{-2\nu}+R^2\sum_{n=0}^{\infty}\lambda_n\sum_{\nu=1}^{\infty}\lambda_{\nu+n}.\xi^{2\nu} \tag{19}$$

or

$$\sum_{\nu=0}^{\infty}A_\nu\sum_{\kappa=0}^{\nu}\lambda^{(\nu)}_{\frac{\nu-\kappa}{2}}.\tau^{\kappa}+\sum_{\nu=0}^{\infty}A_\nu\sum_{\kappa=1}^{\infty}\lambda^{(\nu)}_{\frac{\kappa+\nu}{2}}.\tau^{-\kappa}+\sum_{\nu=0}^{\infty}A_\nu\sum_{\kappa=0}^{\nu}\lambda^{(\nu)}_{\frac{\nu-\kappa}{2}}.\tau^{-\kappa}+\sum_{\nu=0}^{\infty}A_\nu\sum_{\kappa=1}^{\infty}\lambda^{(\nu)}_{\frac{\kappa+\nu}{2}}.\tau^{\kappa}$$

$$=R^2\sum_{n=0}^{\infty}\lambda_n\sum_{\nu=0}^{n}\lambda_{n-\nu}.\tau^{-2\nu}+R^2\sum_{n=0}^{\infty}\lambda_n\sum_{\nu=1}^{\infty}\lambda_{\nu+n}.\tau^{2\nu} \tag{20}$$

Equation (20) can also be written in the following form by changing the sequence in the equation:
It can be written as shown below by changing the sequence in the equation (20)

$$\sum_{\nu=0}^{\infty}A_\nu\sum_{\kappa=0}^{\nu}\lambda^{(\nu)}_{\frac{\nu-\kappa}{2}}.\tau^{\kappa}=\sum_{\kappa=0}^{\infty}\tau^{\kappa}\sum_{\nu=\kappa}^{\infty}A_\nu\lambda^{(\nu)}_{\frac{\nu-\kappa}{2}}=\sum_{\kappa=0}^{\infty}\tau^{\kappa}T_1^{(\kappa)}$$

$$T_1^{(\kappa)}=\sum_{\nu=\kappa}^{\infty}A_\nu\lambda^{(\nu)}_{\frac{\nu-\kappa}{2}}$$

$$\sum_{\nu=0}^{\infty}A_\nu\sum_{\kappa=1}^{\infty}\lambda^{(\nu)}_{\frac{\kappa+\nu}{2}}.\tau^{-\kappa}=\sum_{\kappa=1}^{\infty}\tau^{-\kappa}\sum_{\nu=0}^{\infty}A_\nu\lambda^{(\nu)}_{\frac{\nu+\kappa}{2}}=\sum_{\kappa=1}^{\infty}\tau^{-\kappa}T_2^{(\kappa)}$$

$$T_2(\kappa)=\sum_{\nu=0}^{\infty}A_\nu\lambda^{(\nu)}_{\frac{\kappa+\nu}{2}}$$

$$\sum_{\nu=0}^{\infty}A_\nu\sum_{\kappa=0}^{\nu}\lambda^{(\nu)}_{\frac{\nu-\kappa}{2}}.\tau^{-\kappa}=\sum_{\kappa=0}^{\infty}\tau^{-\kappa}\sum_{\nu=\kappa}^{\infty}A_\nu\lambda^{(\nu)}_{\frac{\nu-\kappa}{2}}=\sum_{\kappa=0}^{\infty}\tau^{-\kappa}T_1(\kappa)$$

$$\sum_{\nu=0}^{\infty}A_\nu\sum_{\kappa=1}^{\infty}\lambda^{(\nu)}_{\frac{\kappa-\nu}{2}}.\tau^{\kappa}=\sum_{\kappa=1}^{\infty}\tau^{\kappa}\sum_{\nu=0}^{\infty}A_\nu\lambda^{(\nu)}_{\frac{\kappa+\nu}{2}}=\sum_{\kappa=1}^{\infty}\tau^{\kappa}T_2(\kappa)$$

$$=\sum_{n=0}^{\infty}\lambda_n\sum_{\nu=0}^{n}\lambda_{n-\nu}.\tau^{-2\nu}=\sum_{\nu=0}^{\infty}\tau^{-2\nu}\sum_{n=\nu}^{\infty}\lambda_n.\lambda_{n-\nu}=\sum_{\nu=0}^{\infty}\tau^{-2\nu}M_1(\nu)$$

$$M_1(\nu)=\sum_{n=\nu}^{\infty}\lambda_n.\lambda_{n-\nu}=\sum_{n=0}^{\infty}\lambda_n\sum_{\nu=1}^{\infty}\lambda_{n+\nu}.\tau^{2\nu}=\sum_{\nu=1}^{\infty}\tau^{2\nu}\sum_{n=0}^{\infty}\lambda_n.\lambda_{n+\nu}=\sum_{\nu=1}^{\infty}\tau^{2\nu}M_2(\nu)$$

$$M_2(\nu)=\sum_{n=0}^{\infty}\lambda_n.\lambda_{n+\nu} \tag{21}$$

These expressions are taken into account in boundary conditions in (20), thereby taking the following form:

$$\sum_{\kappa=0}^{\infty}\tau^{\kappa}T_1(\kappa)+\sum_{\kappa=1}^{\infty}\tau^{-\kappa}T_2(\kappa)+\sum_{\kappa=0}^{\infty}\tau^{-\kappa}T_1(\kappa)+\sum_{\kappa=1}^{\infty}\tau^{\kappa}T_2(\kappa)$$

$$=R^2\sum_{\nu=0}^{\infty}\tau^{-2\nu}M_1(\nu)+R^2\sum_{\nu=1}^{\infty}\tau^{2\nu}M_2(\nu) \qquad (22)$$

In a comparison of the coefficients of τ variable having the same order, the two unlimited linear algebraic equation are found as follows.

$$T_1(\kappa)+T_2(\kappa)=M_1(\frac{\nu}{2})$$

$$T_1(\kappa)+T_2(\kappa)=M_2(\frac{\nu}{2}) \qquad (23)$$

The analysis of this system shows that these systems are identical. Therefore, the unidentified coefficient of a_κ included in expressions $T_1(\kappa)$ and $T_2(\kappa)$ was taken from expression (23). Then, having known the analytic values of $\varphi(z)$ function, τ_{xz} and τ_{yz} tangential stress components were calculated. (Torsional approach $\sigma_x=\sigma_y=\sigma_z=\tau_{xy}=0$).

$$\tau_{xz}-\dot{\iota}\tau_{yz}=\mu\tau i[\varphi'(z)-\bar{z}]$$

Where; μ Lame is a coefficient of creep modulus, τ is the torsional degree $\tau=\frac{M}{D}$, M is the torsional moment, and D is the torsional hardness.

The found solution was verified with the following specific numerical example:

Example: Torsion of a circular beam with two linear radial identical cracks.

In the conformal function, it is assumed that $e_1=e_2$ (a=b, $a_1=b_1$), thus considering a circular beam having two identical linear cracks on a two dimensional Ox axis. Shear stress components τ_{xz} and τ_{yz} take the following values at the most critical points:

If z=e, then e=0,5R

$$\frac{\tau_{xz}}{\mu\tau e}=0.434$$

If z= iR, then e=0,5R

$$\frac{\tau_{yz}}{\mu\tau e}=-0.874$$

The sum of the linear algebraic equations series has been previously explained elsewhere(2). Therefore, it was not focused in this work.

3. Summary

According to the analyzed results, τ_{xz} and τ_{yz} stress components at points $z=\mp e_1, e_2$ (also exceeding the safety stress value) exhibit a local character for maximum stress formation at the end points of the crack. Maximum stresses decline at a great pace where the crack depth is low. These are described in association with the end-point characteristics of the cross section.

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